

# ON A FOURIER PROBLEM FOR EVOLUTION DIFFERENTIAL-FUNCTIONAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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## INTRODUCTION.

The boundary value problems with nonlocal boundary conditions have been treated in many works. Individually, Day [1] while studying quasistatic thermoelasticity posed a model involving parabolic equation with nonlocal boundary condition. This model has been expanded into more general problems [1,2]. The results of these papers concern the boundary value problems with an initial condition.

In the current paper, we consider Fourier Problem (the problem without initial conditions) for evolution differential-functional equations with nonlocal boundary conditions. Note that Fourier Problem for strongly non-linear parabolic equations of the second order with nonlocal boundary conditions of the type of periodicity was investigated in [5].

Let us introduce several concepts and symbols we need later. Let  $D$  be a domain in the space  $\mathbb{R}_{x,t}^{n+1}$ . Denote by  $C^{\alpha,\alpha/2}(\overline{D})$ , where  $\alpha$  is a number from interval  $[0;1]$ , Banach space of real-valued functions, which are continuous in  $\overline{D}$ , if  $\alpha = 0$ , and Hölder continuous functions in  $\overline{D}$  with exponent  $\alpha$ , if  $\alpha > 0$  (see definitions in [8], p.16). Denote by  $C^{2+\alpha,1+\alpha/2}(\overline{D})$  a subspace of space  $C^{\alpha,\alpha/2}(\overline{D})$  which consists of functions  $w$  such that  $\{w_{x_i x_j} \mid \{i,j\} \subset \{1, \dots, n\}\}, w_t\} \subset C^{\alpha,\alpha/2}(\overline{D})$ . The norms in these spaces are denoted by  $\|\cdot\|_{\alpha,\alpha/2}^D$  and  $\|\cdot\|_{2+\alpha,1+\alpha/2}^D$ , respectively. If  $D$  is unbounded domain then denote by  $C_{loc}^{\alpha,\alpha/2}(\overline{D})$ ,  $C_{loc}^{2+\alpha,1+\alpha/2}(\overline{D})$  the spaces of functions defined in  $\overline{D}$  which restrictions on the closure of any bounded subdomain  $D'$  of domain  $D$  belong to  $C^{\alpha,\alpha/2}(\overline{D}')$  and  $C^{2+\alpha,1+\alpha/2}(\overline{D}')$ , respectively ( $\alpha \in [0;1]$ ). Set  $C(\overline{D}) \stackrel{def}{=} C^{0,0}(\overline{D})$ ,  $C_{loc}(\overline{D}) \stackrel{def}{=} C_{loc}^{0,0}(\overline{D})$ . In the case when  $Q$  is conjugation of domain  $D$  and the part of its boundary, we denote by  $C_{loc}^{\alpha,\alpha/2}(Q)$ ,  $C_{loc}^{2+\alpha,1+\alpha/2}(Q)$  the spaces of functions which restrictions on closure of arbitrary bounded subdomain  $D'$  of domain  $D$  such that  $\overline{D'} \subset Q$ , belong to spaces  $C^{\alpha,\alpha/2}(\overline{D'})$  and  $C^{2+\alpha,1+\alpha/2}(\overline{D'})$ , respectively ( $\alpha \in [0;1]$ ).

## 1. STATEMENT OF THE PROBLEM AND FORMULATION OF MAIN RESULTS.

Let  $Q = \Omega \times (-\infty, T]$ ,  $0 < T < +\infty$ ,  $\Omega$  be a bounded domain in  $\mathbb{R}_x^n$  with smooth boundary  $\partial\Omega$ ,  $\Sigma = \partial\Omega \times (-\infty, T]$ .

We consider a problem

$$Pu(x, t) \equiv \frac{\partial u(x, t)}{\partial t} - Lu(x, t) + a(x, t)u(x, t) -$$

$$-f(x, t, u(x, t); u(\cdot, t)) = \hat{f}(x, t), \quad (x, t) \in Q, \quad (1)$$

$$Bu(x, t) \equiv u(x, t) - g(x, t, u(x, t); u(\cdot, t)) = h(x, t), \quad (x, t) \in \Sigma, \quad (2)$$

where

$$Lu(x, t) \equiv \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 u(x, t)}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(x, t) \frac{\partial u(x, t)}{\partial x_i};$$

$f(x, t, \xi; \cdot) : C(\bar{\Omega}) \rightarrow \mathbb{R}$ ,  $(x, t, \xi) \in Q \times \mathbb{R}$ , and  $g(x, t, \eta; \cdot) : C(\bar{\Omega}) \rightarrow \mathbb{R}$ ,  $(x, t, \eta) \in \Sigma \times \mathbb{R}$ , are families of functionals.

Henceforth this problem is called Problem (1),(2).

We impose the following main conditions on the data-in:

- (A1) functions  $a_{ij}, a_i, a$  are continuous in  $Q$ ,  $\{i, j\} \subset \{1, \dots, n\}$ ;  
 (A2)  $a_{ij} = a_{ji}$ ,  $\{i, j\} \subset \{1, \dots, n\}$ , and for arbitrary point  $(x, t) \in Q$  and for all  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$  the following inequality holds

$$\sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \geq \mu(t) \sum_{s=1}^n \xi_s^2,$$

where  $\mu(t) \geq 0$ ,  $t \in (-\infty, T]$ ;

- (A3) for all  $v \in C(\bar{\Omega})$  functions  $f(x, t, \xi; v(\cdot))$ ,  $(x, t, \xi) \in Q \times \mathbb{R}$ ,  $g(x, t, \xi; v(\cdot))$ ,  $(x, t, \xi) \in \Sigma \times \mathbb{R}$ , are continuous, nondecreasing in  $\xi$ , i.e. for arbitrary  $\{\xi^1, \xi^2\} \subset \mathbb{R}$ , such that  $\xi^1 \geq \xi^2$ , the following inequalities hold:

$$f(x, t, \xi^1; v(\cdot)) - f(x, t, \xi^2; v(\cdot)) \geq 0, \quad (x, t) \in Q,$$

$$g(x, t, \xi^1; v(\cdot)) - g(x, t, \xi^2; v(\cdot)) \geq 0, \quad (x, t) \in \Sigma;$$

moreover, these functions are Lipschitz in  $\xi$ , more precisely, there exist functions  $L^f(x, t)$ ,  $(x, t) \in Q$ , and  $L^g(x, t)$ ,  $(x, t) \in \Sigma$ , such that for arbitrary  $\{\xi^1, \xi^2\} \subset \mathbb{R}$  and  $v \in C(\bar{\Omega})$

$$|f(x, t, \xi^1; v(\cdot)) - f(x, t, \xi^2; v(\cdot))| \leq L^f(x, t) |\xi^1 - \xi^2|, \quad (x, t) \in Q,$$

$$|g(x, t, \xi^1; v(\cdot)) - g(x, t, \xi^2; v(\cdot))| \leq L^g(x, t) |\xi^1 - \xi^2|, \quad (x, t) \in \Sigma;$$

- (A4) functionals  $f(x, t, \xi; \cdot) : C(\bar{\Omega}) \rightarrow \mathbb{R}$ ,  $(x, t, \xi) \in Q \times \mathbb{R}$ , and  $g(x, t, \xi; \cdot) : C(\bar{\Omega}) \rightarrow \mathbb{R}$ ,  $(x, t, \xi) \in \Sigma \times \mathbb{R}$ , are Gâteaux differentiable, more precisely, for arbitrary  $v \in C(\bar{\Omega})$  there exist linear and continuous functionals  $f'_c(x, t, \xi; v(\cdot), \cdot) : C(\bar{\Omega}) \rightarrow \mathbb{R}$ ,  $(x, t, \xi) \in Q \times \mathbb{R}$ , and  $g'_c(x, t, \xi; v(\cdot), \cdot) : C(\bar{\Omega}) \rightarrow \mathbb{R}$ ,  $(x, t, \xi) \in \Sigma \times \mathbb{R}$ , such that

$$\left. \frac{d}{ds} f(x, t, \xi; v(\cdot) + sh(\cdot)) \right|_{s=0} = f'_c(x, t, \xi; v(\cdot), h(\cdot)), \quad (x, t, \xi) \in Q \times \mathbb{R},$$



$$\frac{d}{ds}g(x, t, \xi; v(\cdot) + sh(\cdot))|_{s=0} = g'_c(x, t, \xi; v(\cdot), h(\cdot)), \quad (x, t, \xi) \in \Sigma \times \mathbb{R},$$

for all  $h \in C(\overline{\Omega})$ ; the following inequalities hold:

$$f'_c(x, t, \xi; v(\cdot), h(\cdot)) \geq 0, \quad g'_c(x, t, \xi; v(\cdot), h(\cdot)) \geq 0, \quad \text{if } h \geq 0;$$

moreover, assume that

$$\|f'_c(x, t, \xi; v(\cdot), h(\cdot))\| \leq K^f(x, t) \|h\|_{C(\overline{\Omega})}, \quad (x, t, \xi) \in Q \times \mathbb{R}, \quad v \in C(\overline{\Omega}),$$

$$\|g'_c(x, t, \xi; v(\cdot), h(\cdot))\| \leq K^g(x, t) \|h\|_{C(\overline{\Omega})}, \quad (x, t, \xi) \in \Sigma \times \mathbb{R}, \quad v \in C(\overline{\Omega}),$$

where  $K^f(x, t)$ ,  $(x, t) \in Q$ ,  $K^g(x, t)$ ,  $(x, t) \in \Sigma$ , are some functions;

(A5)

$$\inf_{x \in \Omega} (a(x, t) - F(x, t)) \geq a_0(t), \quad t \in (-\infty, T],$$

where  $F(x, t) \stackrel{\text{def}}{=} K^f(x, t) + L^f(x, t)$ ,  $(x, t) \in Q$ , and  $a_0$  is continuous in  $(-\infty, T]$  function;

(A6)

$$G(x, t) < 1, \quad (x, t) \in \Sigma,$$

where  $G(x, t) \stackrel{\text{def}}{=} K^g(x, t) + L^g(x, t)$ ,  $(x, t) \in \Sigma$ ;

(A7)  $\hat{f} \in C_{\text{loc}}(Q)$ ,  $h \in C_{\text{loc}}(\Sigma)$ .

For the convenience of formulating and proving the results, without loss of generality let us make additional assumption

(A0)  $f(x, t, 0, 0) = 0$ ,  $(x, t) \in Q$ ,  $g(x, t, 0, 0) = 0$ ,  $(x, t) \in \Sigma$ .

In the sequel we assume that conditions (A0)-(A7) hold.

**DEFINITION 1.** A function  $u \in C_{\text{loc}}^{2,1}(Q) \cap C_{\text{loc}}(\overline{Q})$  is called a solution of Problem (1),(2) if it satisfies equation (1) and boundary condition (2).

Before formulating of main results of the work we introduce some notations and concepts.

Let  $b$  be an arbitrary continuous in  $(-\infty, T]$  function. We denote by  $V(b)$  a set of continuous differentiable in  $(-\infty, T]$  functions  $\nu$  which satisfy

$$\nu'(t) < b(t), \quad t \in (-\infty, T], \quad \int_{-\infty}^T (b(t) - \nu'(t)) dt = +\infty.$$

Let us note that when  $b(t) = b_0$ ,  $t \in (-\infty, T]$ , where  $b_0 = \text{const}$ , then the set of functions  $\{ct, t \in (-\infty, T] : c \in \mathbb{R}, c < b_0\}$  is subset of  $V(b)$ .

Let  $H$  be one of the sets  $Q$ ,  $\overline{Q}$  or  $\Sigma$ , and  $\nu \in V(a_0)$ . Denote by  $E_\nu(H)$  a set of continuous functions  $q(x, t)$ ,  $(x, t) \in H$ , which satisfy an inequality

$$|q(x, t)| \leq Ke^{-\nu(t)}, \quad (x, t) \in H,$$

where  $K \geq 0$  is a constant which may depend on  $q$ .

We denote by  $\Phi_{x,t;loc}^{\alpha,\alpha/2}$ , where  $\alpha \in (0; 1]$ , a space of families of functionals  $f(x, t, \xi; \cdot) : C(\bar{\Omega}) \rightarrow \mathbb{R}$ ,  $(x, t, \xi) \in Q \times \mathbb{R}$ , such that for arbitrary numbers  $t_0 < T$ ,  $l_1 > 0$ ,  $l_2 > 0$  there exists a constant  $K \geq 0$  which satisfies an inequality

$$|f(x^1, t^1, \xi; v(\cdot)) - f(x^2, t^2, \xi; v(\cdot))| \leq K[|x^1 - x^2|^\alpha + |t^1 - t^2|^{\alpha/2}]$$

for arbitrary  $\{(x^1, t^1), (x^2, t^2)\} \subset \Omega \times (t_0, T]$ , and any  $\xi \in [-l_1, l_1]$  and  $v \in C(\bar{\Omega})$ ,  $\|v\|_{C(\bar{\Omega})} \leq l_2$ .

**THEOREM 1 (AN APRIORI ESTIMATE OF THE SOLUTION).** *Let  $\hat{f}/(a_0 - \nu') \in E_\nu(Q)$  and  $h/(1 - G) \in E_\nu(\Sigma)$  for some  $\nu$  from  $V(a_0)$ . Then the solution of Problem (1), (2)  $u$  from  $E_\nu(\bar{Q})$  satisfies the following estimate:*

$$|u(x, t)| \leq \max \left\{ \sup_{(y, \tau) \in \Sigma} \frac{|h(y, \tau)|e^{\nu(\tau)}}{1 - G(y, \tau)}, \sup_{(y, \tau) \in Q} \frac{|\hat{f}(y, \tau)|e^{\nu(\tau)}}{a_0(\tau) - \nu'(\tau)} \right\} \cdot e^{-\nu(t)} \equiv M_0 e^{-\nu(t)} \quad (3)$$

for all  $(x, t) \in \bar{Q}$ .

**THEOREM 2 (UNIQUENESS OF THE SOLUTION).** *The solution of Problem (1), (2) from class  $E_\nu(\bar{Q})$ , where  $\nu \in V(a_0)$ , is unique.*

For all  $k \in \mathbb{N}$  let us denote by  $Q^k = Q \cap \{(x, t) : t > -k\}$ ,  $\Sigma^k = \Sigma \cap \{(x, t) : t > -k\}$  and define a function  $u_k \in C_{loc}^{2,1}(Q^k) \cap C(\bar{Q}^k)$ , as the solution of Problem:

$$P_k u_k(x, t) = \hat{f}_k(x, t), \quad (x, t) \in Q^k, \quad (1_k)$$

$$B_k u_k(x, t) = h_k(x, t), \quad (x, t) \in \Sigma^k, \quad (2_k)$$

$$u_k(x, -k) = 0, \quad x \in \bar{\Omega}. \quad (3_k)$$

Here

$$P_k w(x, t) \equiv \frac{\partial w(x, t)}{\partial t} - Lw(x, t) + a(x, t)w(x, t) - f_k(x, t, w(x, t); w(\cdot, t))$$

and  $B_k w(x, t) \equiv w(x, t) - g_k(x, t, w(x, t); w(\cdot, t)) \quad \forall w \in C^{2,1}(\bar{Q})$ ,  $f_k(x, t, \xi; \cdot) = \zeta(t + k)f(x, t, \xi; \cdot)$ ,  $(x, t, \xi) \in Q \times \mathbb{R}$ ,  $g_k(x, t, \xi; \cdot) = \zeta(t + k)g(x, t, \xi; \cdot)$ ,  $(x, t, \xi) \in \Sigma \times \mathbb{R}$ ,  $h_k(x, t) = \zeta(t + k)h(x, t)$ ,  $(x, t) \in \Sigma$ ,  $\hat{f}_k(x, t) = \hat{f}(x, t)\zeta(t + k)$ ,  $(x, t) \in \bar{Q}$ , where  $\zeta$  is smooth and monotonic in  $\mathbb{R}$  function such that  $\zeta(t) = 0$  if  $t \leq 1/2$ ,  $\zeta(t) = 1$  if  $t \geq 1$ ,  $k \in \mathbb{N}$ .

**THEOREM 3 (EXISTENCE OF THE SOLUTION).** *Let the following conditions hold for some  $\alpha \in (0; 1]$  and  $\nu \in V(a_0)$ :*

(B1)  $\{a_{ij}, a_i, a\} \subset C_{loc}^{\alpha, \alpha/2}(\bar{Q})$ ,  $\partial a_{ij}/\partial x_s \in C_{loc}(\bar{Q})$ ,  $\{i, j, s\} \subset \{1, \dots, n\}$ ;  $\mu(t) > 0$ ,  $t \in (-\infty, T]$ ;

(B2)  $f \in \Phi_{x,t;loc}^{\alpha, \alpha/2}$ ;

(B3)  $\hat{f} \in C_{loc}^{\alpha, \alpha/2}(\bar{Q})$ ,  $\hat{f}/(a_0 - \nu') \in E_\nu(\bar{Q})$ ;  $h \in C_{loc}^{2+\alpha, 1+\alpha/2}(\bar{Q})$ ,  $h/(1 - G) \in E_\nu(\Sigma)$ ;

(B4) for all  $k \in \mathbb{N}$  Problem (1<sub>k</sub>) - (3<sub>k</sub>) has a solution  $u_k \in C^{2+\alpha, 1+\alpha/2}(\bar{Q}^k)$ .

Then Problem (1), (2) has a solution  $u$  in  $E_\nu(\bar{Q}) \cap C_{loc}^{\alpha, \alpha/2}(\bar{Q}) \cap C_{loc}^{2+\alpha, 1+\alpha/2}(\bar{Q})$ .



COROLLARY. Let conditions (B1)-(B3) of theorem 3 hold, moreover, the following conditions are true: (C1) there exist a family of functionals  $g^*(x, t, \xi; \cdot) : C(\bar{\Omega}) \rightarrow \mathbb{R}$ ,  $(x, t, \xi; \cdot) \in \bar{Q} \times \mathbb{R}$ , and a function  $\eta \in C_0^\infty(\Omega)$ ,  $0 \leq \eta \leq 1$ , such that  $g(x, t, \xi; \cdot) = g^*(x, t, \xi; \cdot)$ ,  $(x, t, \xi) \in \Sigma \times \mathbb{R}$ , and for arbitrary  $(x, t) \in \bar{Q}$  and  $v \in C(\bar{\Omega})$   $g^*(x, t, 0; (\eta v)(\cdot)) = g^*(x, t, v(x); v(\cdot))$ , moreover,  $\tilde{g}(x, t) \stackrel{\text{def}}{=} g^*(x, t, 0; (\eta w)(\cdot, t)) \in C^{2+\alpha, 1+\alpha/2}(\bar{Q})$  for any  $w \in C_{\text{loc}}^{2+\alpha, 1+\alpha/2}(\bar{Q})$ ;

(C2)  $\partial\Omega \in C^{2+\alpha}$ .

Then the statement of Theorem 3 is true.

REMARK 1. The functionals  $g(x, t, \xi; \cdot) : C(\bar{\Omega}) \rightarrow \mathbb{R}$ ,  $(x, t, \xi) \in \Sigma \times \mathbb{R}$ , from Bitsadze-Samarskii boundary condition

$$g(x, t, \xi; v(\cdot)) = \sum_{k=1}^K g_k(x, t) v(\xi_k(x)), \quad (x, t, \xi) \in \Sigma \times \mathbb{R}, \quad v \in C(\bar{\Omega}),$$

satisfy the condition (C1) of corollary if  $K \in \mathbb{N}$ ,  $g_k \in C^{2+\alpha, 1+\alpha/2}(\bar{Q})$ ,  $g_k(x, t) \geq 0$ ,  $(x, t) \in \Sigma$ ,  $k \in \{1, \dots, K\}$ , and functions  $\xi_k$ ,  $k \in \{1, \dots, K\}$ , are defined in  $\bar{\Omega}$  with values in  $\bar{\Omega}_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \delta\}$ ,  $\delta > 0$ ,  $\xi_k \in C^{2+\alpha}(\Omega)$ .

Let  $\Pi_\nu$  be a space of ordered pairs of functions  $(\hat{f}, h)$  such that  $\hat{f} \in C_{\text{loc}}^{\alpha, \alpha/2}(\bar{Q})$ ,  $\hat{f}/(a_0 - \nu') \in E_\nu(\bar{Q})$ ,  $h \in C_{\text{loc}}^{2+\alpha, 1+\alpha/2}(\bar{Q})$ ,  $h/(1 - G) \in E_\nu(\Sigma)$  for  $\nu \in V(a_0)$ . Let us assume that conditions (B1), (B2), (B4) are fulfilled. Then there exists unique solution of Problem (1), (2) in  $E_\nu(\bar{Q})$  for arbitrary  $(\hat{f}, h) \in \Pi_\nu$ , where  $\nu \in V(a_0)$ . In short, we write this as  $u = NZ_\nu(\hat{f}, h)$ .

THEOREM 4 (CONTINUOUS DEPENDENCE ON DATA-IN). Let conditions (B1), (B2), (B4) of Theorem 3 hold. Then for arbitrary  $\varepsilon > 0$  there exists  $\delta > 0$  such that for any  $(\hat{f}_1, h_1), (\hat{f}_2, h_2) \in \Pi_\nu$ , satisfying conditions:

$$\sup_{(x, t) \in Q} \frac{|\hat{f}_1(x, t) - \hat{f}_2(x, t)| e^{\nu(t)}}{a_0(t) - \nu'(t)} < \delta \quad \text{and} \quad \sup_{(x, t) \in \Sigma} \frac{|h_1(x, t) - h_2(x, t)| e^{\nu(t)}}{1 - G(x, t)} < \delta$$

the following inequality holds:

$$\sup_{(x, t) \in Q} |u_1(x, t) - u_2(x, t)| e^{\nu(t)} < \varepsilon,$$

where  $u_i = NZ_\nu(\hat{f}_i, h_i)$ ,  $i \in \{1, 2\}$ .

## 2. AUXILIARY STATEMENTS

Let  $t_0$  be an arbitrary fixed number in  $(-\infty, T)$ . Set  $Q_0 = \Omega \times (t_0, T]$  and  $\Sigma_0 = \partial\Omega \times (t_0, T]$  where  $0 < T < +\infty$ .

LEMMA 1. Let functions  $\tilde{u}, \hat{u} \in C^{2,1}(Q_0) \cap C(\bar{Q}_0)$  fulfil the inequalities

$$P\tilde{u}(x, t) < P\hat{u}(x, t), \quad (x, t) \in Q_0,$$

$$B\tilde{u}(x, t) < B\hat{u}(x, t), \quad (x, t) \in \Sigma_0, \quad \tilde{u}(x, t_0) < \hat{u}(x, t_0), \quad x \in \bar{\Omega}. \quad (4)$$

Then  $\tilde{u}(x, t) < \hat{u}(x, t)$  for all  $(x, t) \in \overline{Q_0}$ .

*Proof.* Let us assume a contradiction. Let  $t^* \in (t_0, T]$  be a maximum value of variable  $t$  such that  $\tilde{u}(x, t) < \hat{u}(x, t)$  for all  $(x, t) \in \overline{Q_0} \cap \{(x, t) : t_0 \leq t < t^*\}$ . Then there exists a point  $x^* \in \bar{\Omega}$  such that  $\tilde{u}(x^*, t^*) = \hat{u}(x^*, t^*)$ .

It follows from the third inequality of (4) that  $t^* > t_0$ . Using (5) we next show that  $(x^*, t^*) \notin \Sigma_0$ . Assuming that  $(x^*, t^*) \in \Sigma_0$  then for the function  $w \stackrel{\text{def}}{=} \tilde{u} - \hat{u}$  by the second inequality of (4) and condition (A4) we have

$$0 = w(x^*, t^*) = g(x^*, t^*, \tilde{u}(x^*, t^*); \tilde{u}(\cdot, t^*)) - g(x^*, t^*, \hat{u}(x^*, t^*); \hat{u}(\cdot, t^*)) + \\ + B\tilde{u}(x^*, t^*) - B\hat{u}(x^*, t^*) < 0,$$

but this contradicts the statement of lemma.

It leads to the fact that the function  $w$  in  $\overline{Q_0} \cap \{(x, t) : t_0 \leq t \leq t^*\}$  takes a maximum value at the point  $(x^*, t^*) \in Q_0$  and it equals zero. Thus, using condition (A2) we have

$$\partial w(x^*, t^*) / \partial t \geq 0, \quad \partial w(x^*, t^*) / \partial x_s = 0, \quad \sum_{i,j=1}^n a_{ij}(x^*, t^*) \partial^2 w(x^*, t^*) / \partial x_i \partial x_j \leq 0.$$

Hence and by condition (A4) it follows

$$P\tilde{u}(x^*, t^*) - P\hat{u}(x^*, t^*) \geq 0,$$

but this contradicts the third inequality of (4).  $\square$

**LEMMA 2.** Assume that all conditions of Lemma 1 are fulfilled and the inequalities (4) are nonstrict. Then  $\tilde{u}(x, t) \leq \hat{u}(x, t)$  for all  $(x, t) \in \overline{Q_0}$ .

*Proof.* Let us consider an auxiliary function  $\hat{u}_\lambda(x, t) = \hat{u}(x, t) + \lambda e^{m^* t}$ ,  $(x, t) \in \overline{Q_0}$ , where  $\lambda > 0$ ,  $m^* + a_0(t) > 0$ ,  $t \in [0, T]$ . Using conditions (A3)-(A5) we obtain

$$P\hat{u}_\lambda(x, t) = P\hat{u}(x, t) + \lambda e^{m^* t} (m^* + a(x, t)) - [f(x, t, \hat{u}(x, t) + \lambda e^{m^* t}; \hat{u}(\cdot, t) + \lambda e^{m^* t}) - \\ - f(x, t, \hat{u}(x, t); \hat{u}(\cdot, t))] > P\hat{u}(x, t), \quad (x, t) \in Q_0.$$

Since  $P\tilde{u}(x, t) \leq P\hat{u}(x, t)$ , and  $P\hat{u}(x, t) < P\hat{u}_\lambda(x, t)$ , then  $P\tilde{u}(x, t) < P\hat{u}_\lambda(x, t)$ ,  $(x, t) \in Q_0$ . It follows from conditions (A3), (A4) and (A6) that

$$B\hat{u}_\lambda(x, t) \geq B\hat{u}(x, t) + \lambda e^{m^* t} (1 - G(x, t)) > B\hat{u}(x, t), \quad (x, t) \in \Sigma_0.$$

Hence, using the fact that  $\tilde{u}(x, t_0) < \hat{u}_\lambda(x, t_0)$ ,  $x \in \bar{\Omega}$ , in view of Lemma 1 we have  $\tilde{u}(x, t) < \hat{u}_\lambda(x, t)$ , if  $(x, t) \in \overline{Q_0}$ ,  $\lambda > 0$ . Since  $\lim_{\lambda \rightarrow 0+} \hat{u}_\lambda(x, t) = \hat{u}(x, t)$ , then  $\tilde{u}(x, t) \leq \hat{u}(x, t)$ ,  $(x, t) \in \overline{Q_0}$ .  $\square$

**LEMMA 3.** Let  $a_0(t) > 0$ ,  $t \in [t_0, T]$ . Then an arbitrary function  $u \in C(\overline{Q_0}) \cap C^{2,1}(Q_0)$  such that  $Pu$  is the bounded in  $Q_0$  function, fulfil an estimate

$$|u(x, t)| \leq \max \left\{ \max_{y \in \bar{\Omega}} |u(y, t_0)|, \sup_{(y, \tau) \in \Sigma_0} \frac{|Bu(y, \tau)|}{1 - G(y, \tau)}, \sup_{(y, \tau) \in Q_0} \frac{|Pu(y, \tau)|}{a_0(\tau)} \right\}, \quad (x, t) \in \overline{Q_0}. \quad (5)$$



*Proof.* Let  $C \stackrel{\text{def}}{=} \max\{\max_{y \in \bar{\Omega}} |u(y, t_0)|, \sup_{(y, \tau) \in \Sigma_0} \frac{|Bu(y, \tau)|}{1-G(y, \tau)}, \sup_{(y, \tau) \in Q_0} \frac{|Pu(y, \tau)|}{a_0(\tau)}\}$ . Consider a function  $\hat{u} \equiv C$ . It follows from conditions (A3)-(A5) and the choice of  $C$  that

$$P\hat{u}(x, t) \geq C(a(x, t) - F(x, t)) \geq C \cdot a_0(t) \geq Pu(x, t), \quad (x, t) \in Q_0. \quad (6)$$

Using condition (A6), in view of definition of function  $\hat{u}$  we obtain

$$B\hat{u}(x, t) \geq C(1 - G(x, t)) \geq Bu(x, t), \quad (x, t) \in \Sigma_0. \quad (7)$$

It is obvious that  $\hat{u}(x, t_0) \geq u(x, t_0)$ ,  $x \in \bar{\Omega}$ . Hence and by (6), (7) and Lemma 2 we obtain  $\hat{u}(x, t) \geq u(x, t)$ ,  $(x, t) \in \bar{Q}_0$ . It can be shown analogously that  $u(x, t) \geq -\hat{u}(x, t)$ ,  $(x, t) \in \bar{Q}_0$ .  $\square$

LEMMA 4. Let  $a_0(t) > 0$ ,  $t \in [t_0, T]$ . Then arbitrary functions  $\{u_1, u_2\} \subset C(\bar{Q}_0) \cap C^{2,1}(Q_0)$  such that  $Pu_1, Pu_2$  are bounded in  $Q_0$  functions, fulfil an inequality

$$|u_1(x, t) - u_2(x, t)| \leq \max \left\{ \max_{y \in \bar{\Omega}} |u_1(y, t_0) - u_2(y, t_0)|, \right. \\ \left. \max_{(y, \tau) \in \Sigma_0} \frac{|Bu_1(y, \tau) - Bu_2(y, \tau)|}{1 - G(y, \tau)}, \max_{(y, \tau) \in Q_0} \frac{|Pu_1(y, \tau) - Pu_2(y, \tau)|}{a_0(\tau)} \right\}, \quad (x, t) \in \bar{Q}_0.$$

*Proof.* Let  $u_1, u_2$  be the same as in the formulation of lemma. Then

$$f(x, t, u_1(x, t); u_1(\cdot, t)) - f(x, t, u_2(x, t); u_2(\cdot, t)) = f(x, t, u_1(x, t); u_1(\cdot, t)) - \\ - f(x, t, u_2(x, t); u_1(\cdot, t)) + f(x, t, u_2(x, t); u_1(\cdot, t)) - f(x, t, u_2(x, t); u_2(\cdot, t)).$$

We have

$$f(x, t, u_1(x, t); u_1(\cdot, t)) - f(x, t, u_2(x, t); u_1(\cdot, t)) = f_*(x, t)u_{1,2}(x, t),$$

where  $f_*(x, t) = \frac{f(x, t, u_1(x, t); u_1(\cdot, t)) - f(x, t, u_2(x, t); u_1(\cdot, t))}{u_1(x, t) - u_2(x, t)}$ , if  $u_1(x, t) \neq u_2(x, t)$ ,

otherwise  $f_*(x, t) = 0$ ;  $u_{1,2}(x, t) \stackrel{\text{def}}{=} u_1(x, t) - u_2(x, t)$ ,  $(x, t) \in \bar{Q}_0$ .

By condition (A4) we obtain

$$f(x, t, u_2(x, t); u_1(\cdot, t)) - f(x, t, u_2(x, t); u_2(\cdot, t)) = f_{**}(x, t; u_{1,2}(\cdot, t)),$$

where  $f_{**}(x, t; \cdot) : C(\bar{\Omega}) \rightarrow \mathbb{R}$ ,  $(x, t) \in Q$ , is a family of linear continuous and nondecreasing functionals.

The preceding facts lead to

$$f(x, t, u_1(x, t); u_1(\cdot, t)) - f(x, t, u_2(x, t); u_2(\cdot, t)) = \tilde{f}(x, t, u_{1,2}(x, t); u_{1,2}(\cdot, t)),$$

where  $\tilde{f}(x, t, \xi; \cdot) : C(\bar{\Omega}) \rightarrow \mathbb{R}$ ,  $(x, t, \xi) \in Q \times \mathbb{R}$ , is a family of functionals with the same properties as the family  $f(x, t, \xi; \cdot) : C(\bar{\Omega}) \rightarrow \mathbb{R}$ ,  $(x, t, \xi) \in Q \times \mathbb{R}$ .

Analogously

$$g(x, t, u_1(x, t); u_1(\cdot, t)) - g(x, t, u_2(x, t); u_2(\cdot, t)) = \tilde{g}(x, t, u_{1,2}(x, t); u_{1,2}(\cdot, t)),$$

where  $\tilde{g}(x, t, \xi; \cdot) : C(\overline{\Omega}) \rightarrow \mathbb{R}$ ,  $(x, t, \xi) \in \Sigma \times \mathbb{R}$ , is a family of functionals with the same properties as  $g(x, t, \xi; \cdot) : C(\overline{\Omega}) \rightarrow \mathbb{R}$ ,  $(x, t, \xi) \in \Sigma \times \mathbb{R}$ .

Denote  $\hat{f}_k(x, t) \stackrel{\text{def}}{=} Pu_k(x, t)$ ,  $(x, t) \in Q_0$ ,  $k \in \{1, 2\}$ ;  $h_k(x, t) = Bu_k(x, t)$ ,  $(x, t) \in \Sigma_0$ ,  $k \in \{1, 2\}$ . Subtracting from equation (1) for  $u_1$  the same equation for  $u_2$  we obtain

$$\begin{aligned} \tilde{P}u_{1,2}(x, t) &\equiv \frac{\partial u_{1,2}(x, t)}{\partial t} - Lu_{1,2}(x, t) + a(x, t)u_{1,2}(x, t) - \\ &- \tilde{f}(x, t, u_{1,2}(x, t); u_{1,2}(\cdot, t)) = \hat{f}_{1,2}(x, t), \quad (x, t) \in Q_0, \end{aligned} \quad (8)$$

where  $\hat{f}_{1,2}(x, t) \stackrel{\text{def}}{=} \hat{f}_1(x, t) - \hat{f}_2(x, t)$ ,  $(x, t) \in Q_0$ .

Analogously

$$\tilde{B}u_{1,2}(x, t) \equiv u_{1,2}(x, t) - \tilde{g}(x, t, u_{1,2}(x, t); u_{1,2}(\cdot, t)) = h_{1,2}(x, t), \quad (x, t) \in \Sigma_0, \quad (9)$$

$$u_{1,2}(x, t_0) = u^1(x, t_0) - u^2(x, t_0), \quad x \in \Omega, \quad (10)$$

where  $h_{1,2}(x, t) \stackrel{\text{def}}{=} h_1(x, t) - h_2(x, t)$ ,  $(x, t) \in \Sigma_0$ .

Using Lemma 3 and (8)-(10), we complete the proof.  $\square$

LEMMA 5. Let  $\nu \in V(a_0)$  and functions  $\{u_1, u_2\} \subset C_{\text{loc}}^{2,1}(Q) \cap C_{\text{loc}}(\overline{Q})$  are such that  $u_1 - u_2 \in E_\nu(\overline{Q})$ ,  $(Pu_1 - Pu_2)/(a_0 - \nu') \in E_\nu(Q)$ ,  $(Bu_1 - Bu_2)/(1 - G) \in E_\nu(\Sigma)$ . Then

$$\begin{aligned} |u_1(x, t) - u_2(x, t)| &\leq \max \left\{ \sup_{(y, \tau) \in \Sigma} \frac{|Bu_1(y, \tau) - Bu_2(y, \tau)|e^{\nu(\tau)}}{1 - G(y, \tau)}, \right. \\ &\left. \sup_{(y, \tau) \in Q} \frac{|Pu_1(y, \tau) - Pu_2(y, \tau)|e^{\nu(\tau)}}{a_0(\tau) - \nu'(\tau)} \right\} e^{-\nu(t)}, \quad (x, t) \in \overline{Q}. \end{aligned} \quad (11)$$

*Proof.* First we consider the case  $\nu = 0 \in V(a_0)$  and use the ideas of paper [7]. Denote  $\hat{f}_k(x, t) \stackrel{\text{def}}{=} Pu_k(x, t)$ ,  $(x, t) \in Q$ ,  $k \in \{1, 2\}$ ;  $h_k(x, t) = Bu_k(x, t)$ ,  $(x, t) \in \Sigma$ ,  $k \in \{1, 2\}$ . Let  $\lambda(t) = \int_T^t a_0(\tau) d\tau$ ,  $\tau \in (-\infty, T]$ , and  $\gamma \in (0; 1)$ . Let us multiply equation (1) and condition (2) for  $u_1$  and  $u_2$  by  $e^{\gamma\lambda(t)}$ . After simple transformations we obtain

$$\begin{aligned} P^\gamma \tilde{u}_{\gamma,k}(x, t) &\equiv \frac{\partial \tilde{u}_{\gamma,k}(x, t)}{\partial t} - L\tilde{u}_{\gamma,k}(x, t) + (a(x, t) - \gamma a_0(t))\tilde{u}_{\gamma,k}(x, t) - \\ &- f(x, t, \tilde{u}_{\gamma,k}(x, t)e^{-\gamma\lambda(t)}; \tilde{u}_{\gamma,k}(\cdot, t)e^{-\gamma\lambda(t)})e^{\gamma\lambda(t)} = \hat{f}_k(x, t)e^{\gamma\lambda(t)}, \quad (x, t) \in Q, \end{aligned} \quad (12)$$

$$\begin{aligned} B^\gamma \tilde{u}_{\gamma,k}(x, t) &\equiv \tilde{u}_{\gamma,k}(x, t) - g(x, t, \tilde{u}_{\gamma,k}(x, t)e^{-\gamma\lambda(t)}; \tilde{u}_{\gamma,k}(\cdot, t)e^{-\gamma\lambda(t)})e^{\gamma\lambda(t)} = \\ &= h_{1,2}(x, t)e^{\gamma\lambda(t)}, \quad (x, t) \in \Sigma, \quad k \in \{1, 2\}, \end{aligned} \quad (13)$$

where  $\tilde{u}_{\gamma,k}(x, t) = u_k(x, t)e^{\gamma\lambda(t)}$ ,  $(x, t) \in Q$ ,  $k \in \{1, 2\}$ .

Let  $t_*$  be an arbitrary negative number,  $Q_* = \Omega \times (t_*, T]$ ,  $\Sigma_* = \partial\Omega \times (t_*, T]$ . It is easy to see that coefficients of the differential operators  $P^\gamma$  and  $B^\gamma$  fulfil the conditions



similar to conditions (A0)-(A7) for the coefficients of the operators  $P$  and  $B$  with  $a_0(1-\gamma)$  in place of  $a_0$ . Thus, in view of Lemma 4, by (12) and (13) we obtain

$$|\tilde{u}_{\gamma,1,2}(x,t)| \leq \max\{e^{\gamma\lambda(t_*)} \cdot \max_{y \in \bar{\Omega}} |u_{1,2}(y,t_*)|, e^{\gamma\lambda(T)} \cdot \sup_{(y,\tau) \in \Sigma_{\text{Sigma}}} \frac{|h_{1,2}(y,\tau)|}{1-G(y,\tau)}, \\ \frac{e^{\gamma\lambda(T)}}{1-\gamma} \cdot \sup_{(y,\tau) \in Q_*} \frac{|\hat{f}_{1,2}(y,\tau)|}{a_0(\tau)}\}, \quad (x,t) \in \bar{Q}_*, \quad (14)$$

where  $\tilde{u}_{\gamma,1,2} \stackrel{\text{def}}{=} \tilde{u}_{\gamma,1}(x,t) - \tilde{u}_{\gamma,2}(x,t)$ ,  $(x,t) \in \bar{Q}$ ;  $\hat{f}_{1,2}(x,t) \stackrel{\text{def}}{=} \hat{f}_1(x,t) - \hat{f}_2(x,t)$ ,  $(x,t) \in Q$ ;  $h_{1,2}(x,t) \stackrel{\text{def}}{=} h_1(x,t) - h_2(x,t)$ ,  $(x,t) \in \Sigma$ .

Since  $u_1 - u_2 \in E_0(\bar{Q})$  then  $|u_{1,2}(x,t)| \leq C_1$  for all  $(x,t) \in \bar{Q}$ , where  $C_1 \geq 0$  is a constant. Because  $e^{\gamma\lambda(t_*)} \rightarrow 0$  as  $t_* \rightarrow -\infty$  implies  $e^{\gamma\lambda(t_*)} \cdot \max_{y \in \bar{\Omega}} |u_{1,2}(y,t_*)| \rightarrow 0$  as  $t_* \rightarrow -\infty$ . Hence, letting first  $t_* \rightarrow -\infty$  in (14) and after letting  $\gamma \rightarrow 0+$  we obtain (11) for  $\nu = 0$ .

Let  $\nu \in V(a_0)$ ,  $\nu \neq 0$ . Let us multiply equation (1) and condition (2) for  $u_1$  and  $u_2$  by  $e^{\nu(t)}$ . After simple transformations we obtain (see (18),(19))

$$P_\nu \hat{u}_k(x,t) = \hat{f}_k(x,t)e^{\nu(t)}, \quad (x,t) \in Q, \quad k \in \{1,2\},$$

$$B_\nu \hat{u}_k(x,t) = h_k(x,t)e^{\nu(t)}, \quad (x,t) \in \Sigma, \quad k \in \{1,2\},$$

where  $\hat{u}_k(x,t) = u_k(x,t)e^{\nu(t)}$ ,  $(x,t) \in \bar{Q}$ ,  $k \in \{1,2\}$ ;  $P_\nu v(x,t) \equiv \partial v(x,t)/\partial t - Lv(x,t) + (a(x,t) - \nu'(t))v(x,t) - f(x,t,v(x,t)e^{-\nu(t)}; v(\cdot,t)e^{-\nu(t)})e^{\nu(t)}$ ,  $(x,t) \in Q$ ,  $B_\nu v(x,t) \equiv v(x,t) - g(x,t,v(x,t)e^{-\nu(t)}; v(\cdot,t)e^{-\nu(t)})e^{\nu(t)}$ ,  $(x,t) \in \Sigma$ ,  $v \in C_{\text{loc}}^{2,1}(Q)$ . It is easy to see that coefficients of the differential operators  $P_\nu$  and  $B_\nu$  fulfill the conditions analogous to conditions (A1)-(A6) for the coefficients of the operators  $P$  and  $B$  with  $a_0 - \nu'$  in place of  $a_0$ . It is obvious that  $\hat{u}_{1,2} \in E_0(\bar{Q})$ . Hence, based on the proof of lemma for  $\nu = 0$  we complete the proof.  $\square$

### 3. PROOF OF BASIC RESULTS

*Proof of Theorem 1.* To obtain an apriori estimate of the solution of Problem (1),(2) it suffices to set  $u_1 = u$ ,  $u_2 = 0$  and use Lemma 5.

*Proof of Theorem 2* As a consequence of Lemma 5, we have the uniqueness of the solution.

*Proof of Theorem 3.* Consider first the case  $\nu = 0 \in V(a_0)$ . Let for all  $k \in \mathbb{N}$   $u_k$  be a solution of Problem (1<sub>k</sub>) - (3<sub>k</sub>).

Based on Lemma 4 we have  $u_k(x,t) = 0$  for all  $(x,t) \in \bar{\Omega} \times [-k, -k + 1/2]$ . Let us extend the function  $u_k$  by zero on  $\bar{Q} \setminus \bar{Q}^k$  and denote these extensions again by  $u_k$  ( $k \in \mathbb{N}$ ). It is obvious that  $u_k \in C^{2+\alpha, 1+\alpha/2}(\bar{Q})$  and  $u_k$  is a solution of Problem (1),(2) with  $f_k, \hat{f}_k, g_k, h_k$  in place of  $f, \hat{f}, g, h$ , respectively, we denote this by  $u_k = NZ_0^k(\hat{f}_k, h_k)$ ,  $k \in \{1, 2, \dots\}$ . In view of Theorem 1 we have

$$|u_k(x,t)| \leq M_0, \quad (x,t) \in \bar{Q}, \quad k \in \mathbb{N}. \quad (15)$$

We show that the restrictions of terms of sequence  $\{u_k\}$  on the closure  $\bar{Q}'$  of an arbitrary bounded subdomain  $Q'$  of domain  $Q$  is the fundamental sequence in  $C(\bar{Q}')$ .



Let for arbitrary  $\{k, l\} \subset \mathbb{N}$   $u_k = NZ_0^k(\hat{f}_k, h_k)$ ,  $u_l = NZ_0^k(\hat{f}_l, h_l)$ . Set  $\lambda(t) \stackrel{\text{def}}{=} \int_T^t a_0(\tau) d\tau$ ,  $t \in (-\infty, T]$ , and multiply the equalities (1<sub>m</sub>) and (2<sub>m</sub>),  $m \in \{k, l\}$ , by  $e^{\frac{1}{2}\lambda(t)}$ . After simple transformations we obtain

$$\begin{aligned} & \frac{\partial \tilde{u}_m(x, t)}{\partial t} - L\tilde{u}_m(x, t) + (a(x, t) - \frac{1}{2}a_0(t))\tilde{u}_m(x, t) - \\ & - f_m(x, t, \tilde{u}_m(x, t)e^{-\frac{1}{2}\lambda(t)}; \tilde{u}_m(\cdot, t)e^{-\frac{1}{2}\lambda(t)})e^{\frac{1}{2}\lambda(t)} = \hat{f}_m(x, t)e^{\frac{1}{2}\lambda(t)}, \quad (x, t) \in Q, \end{aligned} \quad (16)$$

$$\begin{aligned} & \tilde{u}_m(x, t) - g_m(x, t, \tilde{u}_m(x, t)e^{-\frac{1}{2}\lambda(t)}; \tilde{u}_m(\cdot, t)e^{-\frac{1}{2}\lambda(t)})e^{\frac{1}{2}\lambda(t)} = \\ & = h_m(x, t)e^{\frac{1}{2}\lambda(t)}, \quad (x, t) \in \Sigma, \end{aligned} \quad (17)$$

where  $\tilde{u}_m(x, t) = u_m(x, t)e^{\frac{1}{2}\lambda(t)}$ ,  $(x, t) \in \bar{Q}$ ,  $m \in \{k, l\}$ .

Let us show that for an arbitrary fixed natural number  $m$  the restrictions of terms of the sequence  $\{u_k\}$  on  $\bar{Q}^m$  compose fundamental sequence in  $C(\bar{Q}^m)$ . We take an arbitrary value  $\varepsilon > 0$  and fix it. Let  $k_0 \in \mathbb{N}$  is such that  $k_0 > m$  and  $2M_0 \cdot e^{\frac{1}{2}\lambda(-k_0)} < \varepsilon$ . Let  $k$  and  $l$  be arbitrary natural numbers greater than  $k_0$  (constant  $M_0$  is from (3)). It leads to  $\hat{f}_k(x, t) - \hat{f}_l(x, t) = 0$ ,  $(x, t) \in Q^{k_0}$ , and  $h_k(x, t) - h_l(x, t) = 0$ ,  $(x, t) \in \Sigma^{k_0}$ .

Consider the restrictions of equalities (16) and (17) on  $Q^{k_0}$ . Based on Lemma 4 with  $\frac{1}{2}a_0(t)$  in place of  $a_0(t)$  we obtain

$$|\tilde{u}_k(x, t) - \tilde{u}_l(x, t)| \leq \max_{y \in \bar{\Omega}} |\tilde{u}_k(y, -k_0) - \tilde{u}_l(y, -k_0)|, \quad (x, t) \in \bar{Q}^{k_0}. \quad (18)$$

By (15) we have  $\max_{y \in \bar{\Omega}} |\tilde{u}_k(y, -k_0) - \tilde{u}_l(y, -k_0)| \leq 2M_0 \cdot e^{\frac{1}{2}\lambda(-k_0)} < \varepsilon$ . This and (18) lead to  $\max_{(x, t) \in \bar{Q}^{k_0}} |\tilde{u}_k(x, t) - \tilde{u}_l(x, t)| < \varepsilon$  for any  $k, l \geq k_0$ . Hence  $\max_{(x, t) \in \bar{Q}^m} |\tilde{u}_{k,l}(x, t)| < \varepsilon$ .

Therefore we have that the restrictions of the sequence  $\{\tilde{u}_k\}$  where  $\tilde{u}_k(x, t) = u_k(x, t)e^{\frac{1}{2}\lambda(t)}$ ,  $(x, t) \in \bar{Q}$ , on the set  $\bar{Q}^m$ , where  $m$  is an arbitrary fixed natural number, is fundamental sequence in  $C(\bar{Q}^m)$ . Thus, there exists function  $u \in C_{\text{loc}}(\bar{Q})$  such that  $u_k \rightarrow u$  uniformly as  $k \rightarrow \infty$  on an arbitrary compact from  $Q$ . In view of (15) we have  $|u(x, t)| \leq M_0$ ,  $(x, t) \in \bar{Q}$ .

Let us show that  $u$  is a solution of Problem (1),(2). Since for any  $k \in \mathbb{N}$   $u_k = NZ_0^k(\hat{f}_k, h_k)$ , we have

$$\begin{aligned} & \frac{\partial u_k(x, t)}{\partial t} - Lu_k(x, t) + a(x, t)u_k(x, t) = \\ & = f_k(x, t, u_k(x, t); u_k(\cdot, t)) + \hat{f}_k(x, t), \quad (x, t) \in Q. \end{aligned} \quad (19_k)$$

Let  $\{\Omega_m\}_{m=1}^{\infty}$  be a sequence of domains  $\Omega_m \subset \Omega$ ,  $m \in \mathbb{N}$  such that  $\Omega_1 \subset \Omega_2 \subset \dots \subset \Omega_m$  subset...,  $\bigcup_{m \in \mathbb{N}} \Omega_m = \Omega$ ,  $\text{dist}(\Omega_m, \partial\Omega_{m+1}) > 0$ ,  $m \in \mathbb{N}$ , ( $\text{dist}(A, B)$  is a distance between the sets  $A$  and  $B$ ). Set  $Q_{(m)} = \Omega_m \times (-m, T]$ ,  $m \in \mathbb{N}$ .



Let  $m$  be an arbitrary fixed natural number. By  $(19_k)$  and Theorem 10.1 of monograph [8; pp.238-239] we obtain  $\|u_k\|_{\alpha, \alpha/2}^{Q(m+1)} \leq C_2$ ,  $k \in \mathbb{N}$ , where  $C_2 > 0$  is a constant which does not depend on  $k$ .

Using this,  $(19_k)$ , conditions of Theorem 3, in view of Theorem 10.1 of monograph [8; p.400] we obtain  $\|u_k\|_{2+\alpha, 1+\alpha/2}^{Q(m)} \leq C_3$ , where  $C_3 \geq 0$  is a constant which does not depend on  $k$ .

Hence, and by well-known properties of space  $C^{2+\alpha, 1+\alpha/2}(\overline{Q(m)})$  it follows that for an arbitrary fixed number  $\gamma \in (0, \alpha)$  there exists a subsequence of sequence  $\{u_k\}_{k=1}^\infty$  which tends to  $u$  in  $C^{2+\gamma, 1+\gamma/2}(\overline{Q(m)})$ , the restrictions of  $u$  on  $\overline{Q(m)}$  belong to the space  $C^{2+\alpha, 1+\alpha/2}(\overline{Q(m)})$ . Hence, using diagonal process, we conclude that there exists a subsequence  $\{u_{k_j}\}_{j=1}^\infty$  of sequence  $\{u_k\}_{k=1}^\infty$  such that for an arbitrary bounded domain  $Q' \subset Q$  located on the positive distance from  $\Sigma$  the restrictions  $u_{k_j}$  on  $\overline{Q'}$ ,  $j \in \mathbb{N}$ , compose a sequence which converges to the restriction of  $u$  on  $\overline{Q'}$  in  $C^{2+\gamma, 1+\gamma/2}(\overline{Q'})$ , and  $u \in C_{loc}^{2+\alpha, 1+\alpha/2}(Q)$ . Hence and by  $(1_k)$  we obtain that  $u$  is a solution of equation (1). The fulfilment of condition (2) follows from  $(2_k)$  and uniform convergence of  $\{u_k\}$  in  $\overline{Q^m}$  for an arbitrary fixed natural number  $m$ . Thus the proof of Theorem 3 is complete in the case of  $\nu = 0 \in V(a_0)$ .

Let  $\nu \in V(a_0)$ ,  $\nu \neq 0$ . Problem (1),(2) can be rewritten

$$\begin{aligned} & \frac{\partial \hat{u}(x, t)}{\partial t} - L\hat{u}(x, t) + (a(x, t) - \nu'(t))\hat{u}(x, t) - \\ & - f(x, t, \hat{u}(x, t)e^{-\nu(t)}; \hat{u}(\cdot, t)e^{-\nu(t)})e^{\nu(t)} = \hat{f}(x, t)e^{\nu(t)}, \quad (x, t) \in Q, \end{aligned} \quad (20)$$

$$\hat{u}(x, t) - g(x, t, \hat{u}(x, t)e^{-\nu(t)}; \hat{u}(\cdot, t)e^{-\nu(t)})e^{\nu(t)} = h(x, t)e^{\nu(t)}, \quad (x, t) \in \Sigma, \quad (21)$$

where  $\hat{u}(x, t) = u(x, t)e^{\nu(t)}$ ,  $(x, t) \in \overline{Q}$ .

Problem (20),(21) is similar to Problem (1),(2) with  $a_0 - \nu'$  in place of  $a_0$  in the case of  $\nu = 0 \in V(a_0)$ . This completes the proof of Theorem 3.  $\square$

*Proof of Corollary.* For the sake of simplicity without loss of generality we consider only the case of  $k = 1$ , more precisely, we consider a problem

$$P_1 u_1(x, t) = \hat{f}_1(x, t), \quad (x, t) \in Q^1, \quad (22)$$

$$B_1 u_1(x, t) = h_1(x, t), \quad (x, t) \in \Sigma^1, \quad (23)$$

$$u_1(x, -1) = 0, \quad x \in \Omega. \quad (24)$$

Further we use the arguments similar to that in [4].

Without loss of generality we take  $a_0(t) > 0$ ,  $t \in [-1, T]$ . Let

$$v_0(x, t) = C, \quad (x, t) \in \overline{Q^1},$$

where

$$C \stackrel{\text{def}}{=} \max \left\{ \sup_{(y, \tau) \in \Sigma^1} \frac{|h_1(y, \tau)|}{1 - G(y, \tau)}, \sup_{(y, \tau) \in Q^1} \frac{|\hat{f}_1(y, \tau)|}{a_0(\tau)} \right\}.$$



Define the sequence of functions  $\{v_p\}_{p=0}^\infty$  by the rule: if the function  $v_{p-1} \in C^{2+\alpha, 1+\alpha/2}(\overline{Q^1})$  is known then a function  $v_p \in C^{2+\alpha, 1+\alpha/2}(\overline{Q^1})$  is a solution of problem

$$\begin{aligned} \hat{P}v_p(x, t) &\equiv \frac{\partial v_p(x, t)}{\partial t} - Lv_p(x, t) + a(x, t)v_p(x, t) = \\ &= f_1(x, t, v_{p-1}(x, t); v_{p-1}(\cdot, t)) + \hat{f}_1(x, t), \quad (x, t) \in Q^1, \end{aligned} \quad (25)$$

$$v_p(x, t) = g_1^*(x, t, 0; (\eta v_{p-1})(\cdot, t)) + h_1(x, t), \quad (x, t) \in \Sigma^1, \quad (26)$$

$$v_p(x, -1) = 0, \quad x \in \Omega, \quad (27)$$

where function  $\eta \in C_0^\infty(\Omega)$  is from condition (C1) of corollary,  $g_1^*(x, t, \xi; \cdot) = \xi(t + 1)g^*(x, t, \xi; \cdot)$ ,  $(x, t, \xi) \in \overline{Q} \times \mathbb{R}$ .

It follows from Theorem 6.1 of monograph [8; p.513], that a sequence  $\{v_p\}$  is correctly defined, that is, for arbitrary  $p \in \mathbb{N}$  function  $v_p \in C^{2+\alpha, 1+\alpha/2}(\overline{Q^1})$  is uniquely found.

Further, using our assumptions and the results of monograph [8] it can be shown that sequence  $\{v^p\}$  converges to the solution of Problem (22)-(24).

□

*Proof of Theorem 4.* Let  $\varepsilon > 0$  be an arbitrary number and  $(\hat{f}_1, h_1), (\hat{f}_2, h_2) \in \Pi_\nu$  such that  $\sup_{(y, \tau) \in Q} \frac{|\hat{f}_1(y, \tau) - \hat{f}_2(y, \tau)|e^{\nu\tau}}{a_0(\tau) - \nu'(\tau)} < \varepsilon/3$ ,  $\sup_{(y, \tau) \in \Sigma} \frac{|h_1(y, \tau) - h_2(y, \tau)|e^{\nu\tau}}{1 - K(y, \tau)} < \varepsilon/3$ , and

$u_i = NZ_\nu(\hat{f}_i, h_i)$ ,  $i \in \{1, 2\}$ . Hence, in view of Lemma 5 we complete the proof of Theorem 4. □

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